Polynomials

Roots and Discriminants

Find Root

$$Ax^2 + 2Bx + C = 0$$

$$Ax^3 + 3Bx^2 + 3Cx + D = 0$$

$$Ax^4 + 4Bx^3 + 6Cx^2 + 4Dx + E = 0$$

Step 1) Translate Parameter

$$x = \hat{x} - B/A$$

Find Root

$$A\hat{x}^2 + \hat{C} = 0$$

$$A\hat{x}^3 + 3\hat{C}\hat{x} + \hat{D} = 0$$

$$A\hat{x}^4 + 6\hat{C}\hat{x}^2 + 4\hat{D}\hat{x} + \hat{E} = 0$$

Sum of Roots = 0

Step 2) Solve simpler Polynomials

Step 3) Transform Back

$$x = \hat{x} - B/A$$

Homogeneous Polynomials

$$Ax^2 + 2Bxw + Cw^2 = 0$$

$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0$$

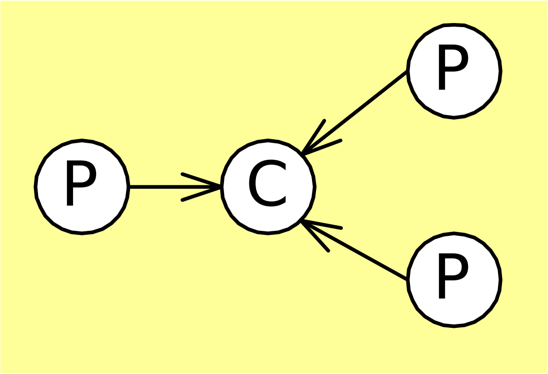
$$Ax^4 + 4Bx^3 + 6Cx^2w + 4Dxw^2 + Ew^3 = 0$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{w} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{B}{A} & 1 \end{bmatrix}$$

Solving Homogeneous Cubic Polynomials

$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3 = 0$$

$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0$$

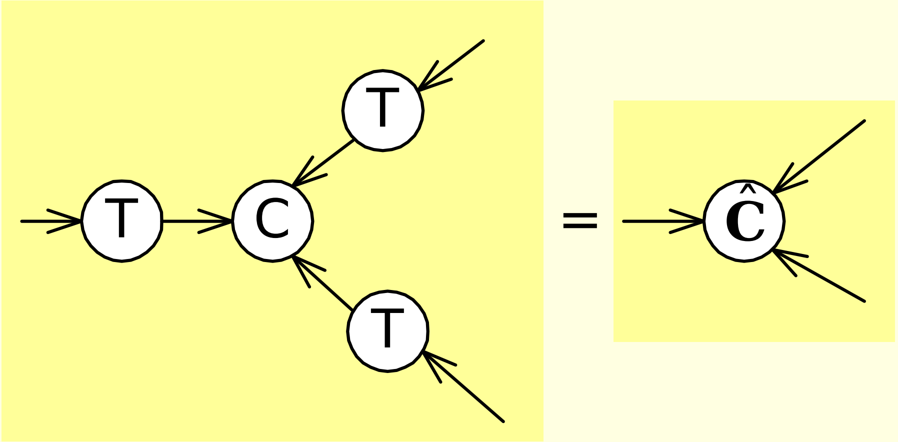


$= 0$

General Homogeneous Parameter Transform

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{w} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

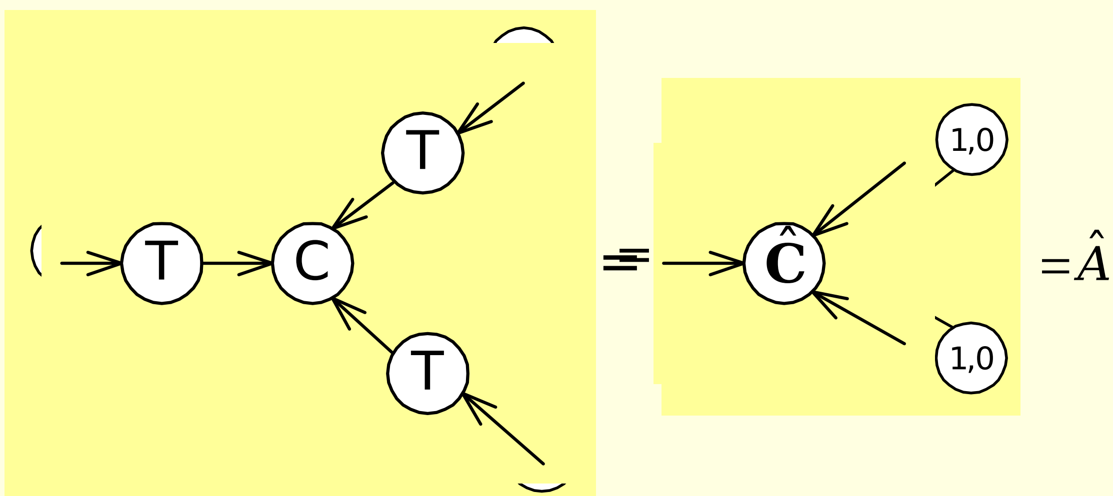
$$\mathbf{p} = \hat{\mathbf{p}}\mathbf{T}$$



Elements of Transformed C

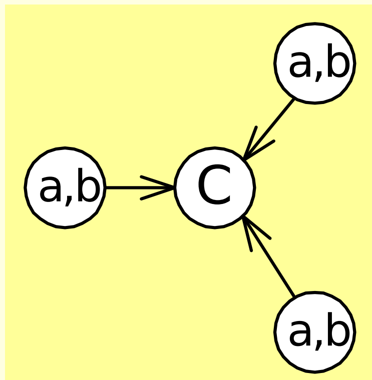
$$\mathbf{T}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix}\end{bmatrix} \rightarrow \begin{bmatrix}\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{B} & \hat{C} \end{bmatrix} & \begin{bmatrix} \hat{B} & \hat{C} \\ \hat{C} & \hat{D} \end{bmatrix}\end{bmatrix}$$

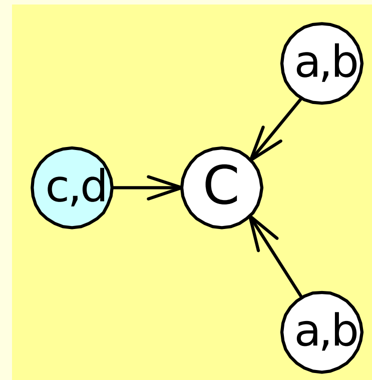


Elements of Transformed Cubic

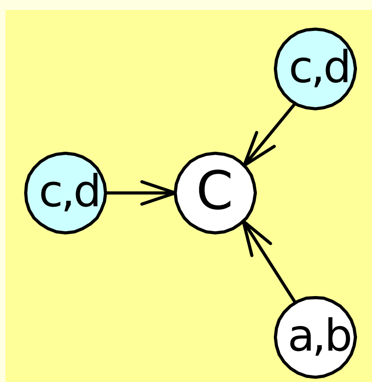
$\hat{A} =$



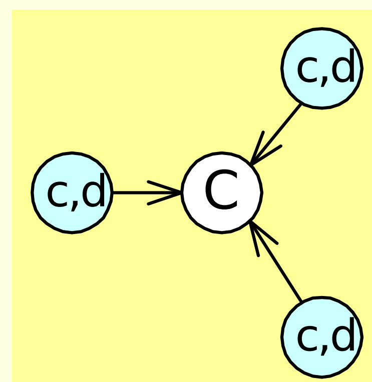
$\hat{B} =$



$\hat{C} =$

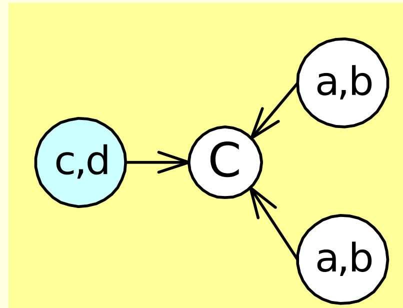


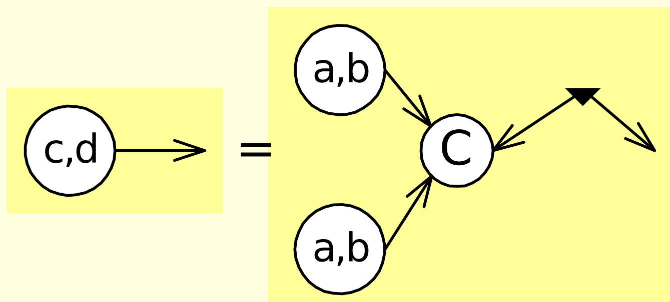
$\hat{D} =$



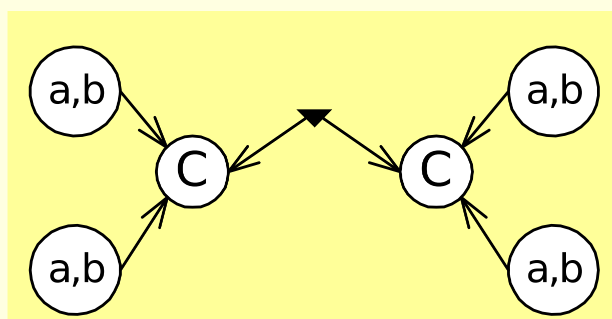
Make B^ zero

$\hat{B} =$

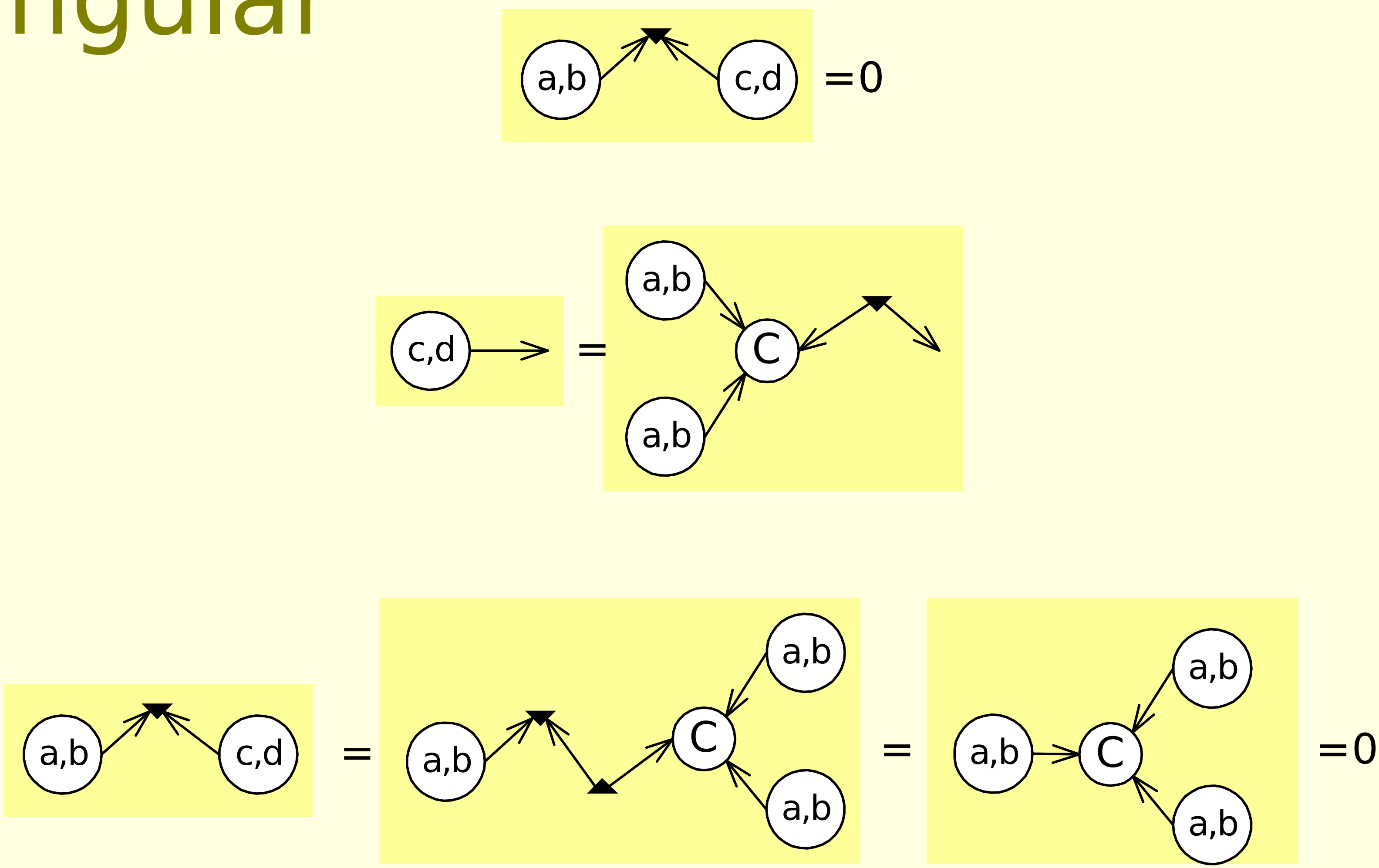




$\hat{B} =$

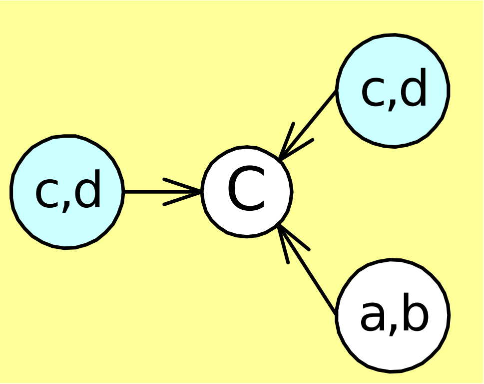


When is Transform Singular



Evaluate C^

$\hat{C} =$



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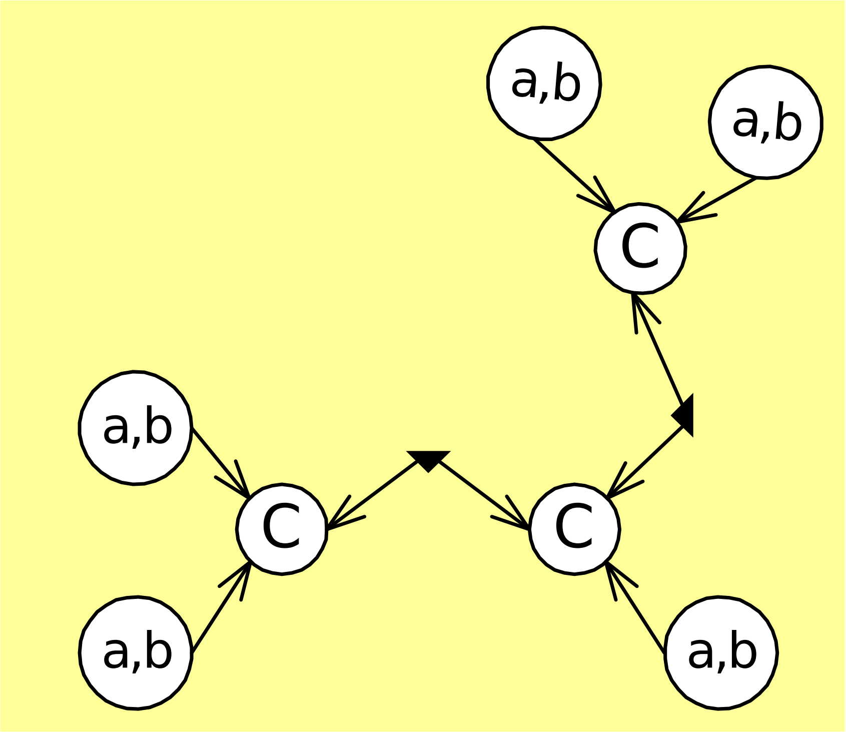
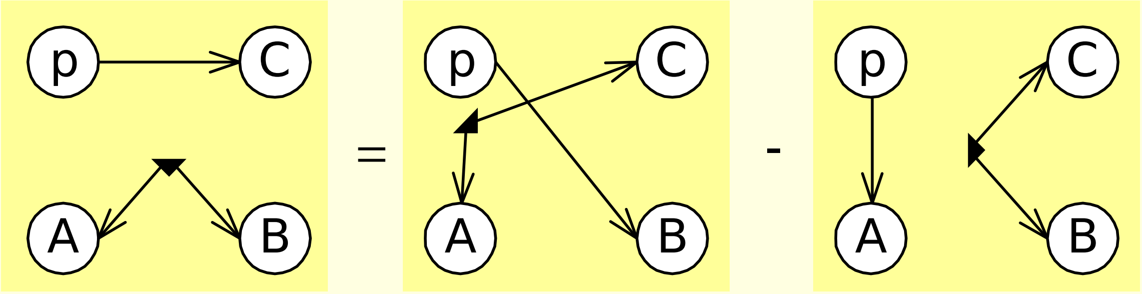
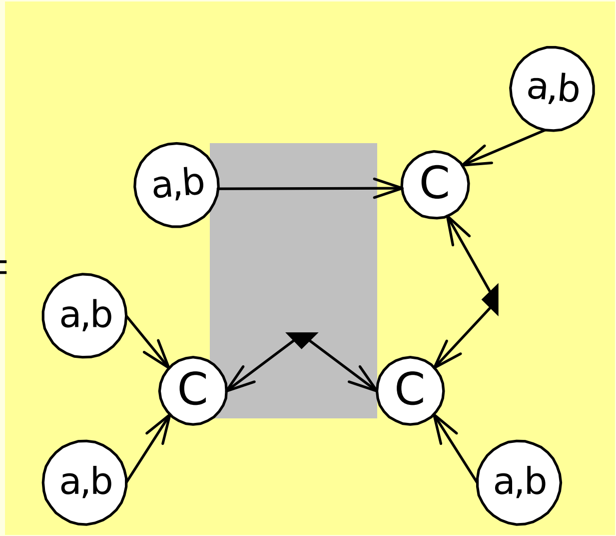


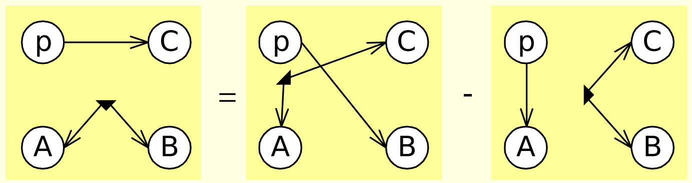
Apply variant of epsilon-delta



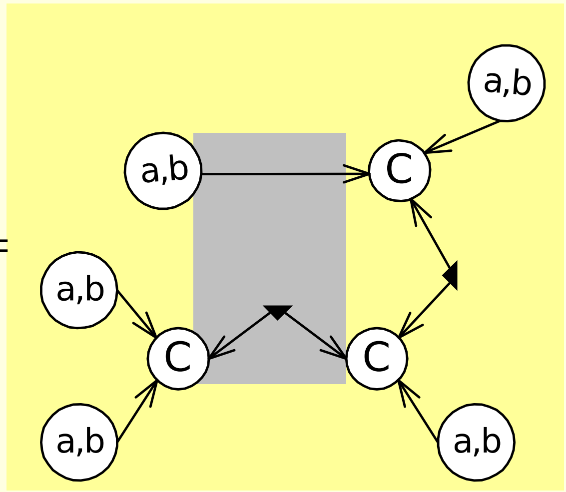
$\hat{C} =$



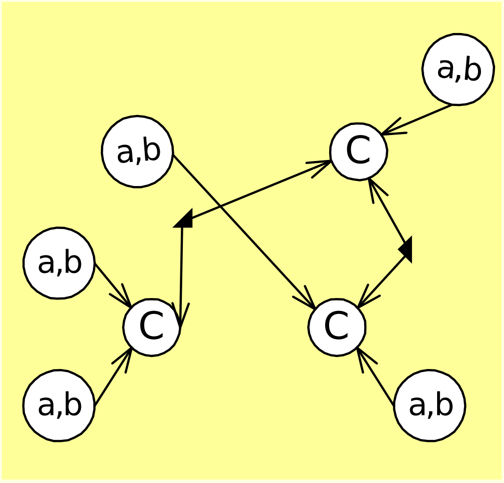
Apply


p
C
A
B
=
p
C
A
B
-
p
C
A
B

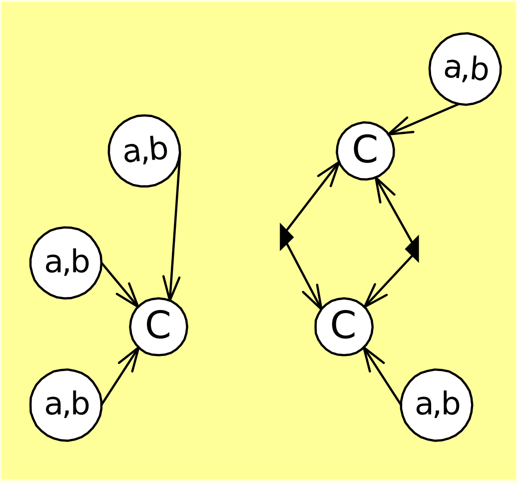
$\hat{C} =$


a,b
a,b
C
a,b
C
C
a,b
a,b

=

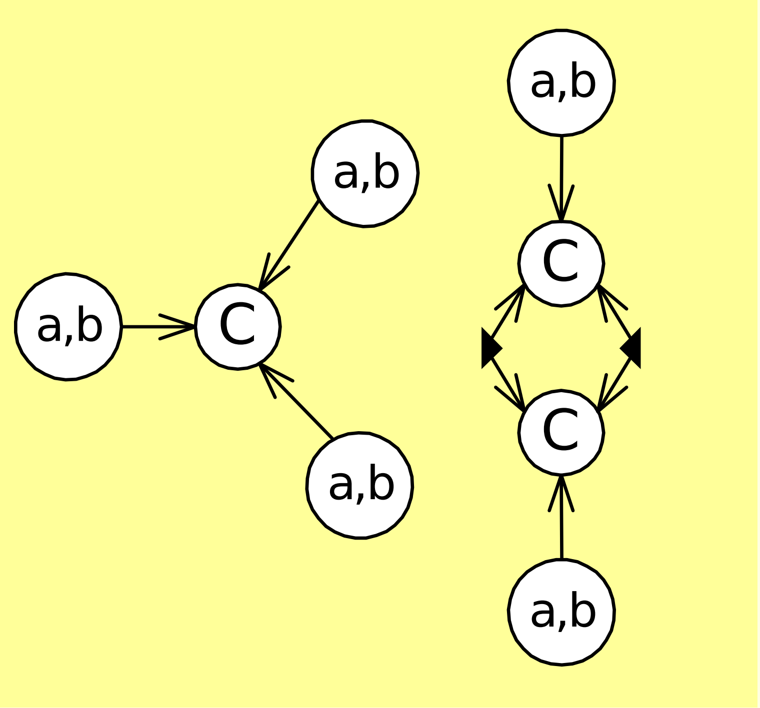

a,b
a,b
C
a,b
C
C
a,b
a,b

-

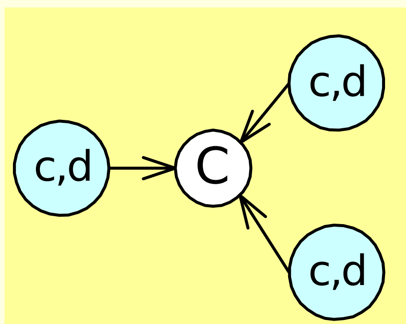

a,b
a,b
C
a,b
C
C
a,b
a,b

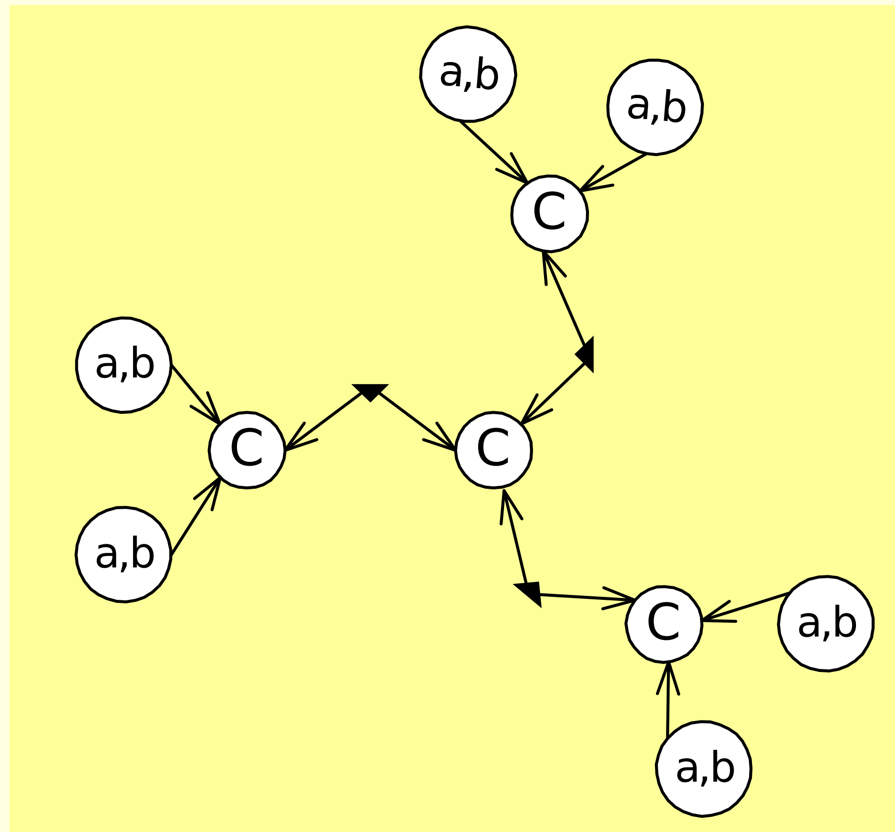
Final C^

$$\hat{C} = -\frac{1}{2}$$



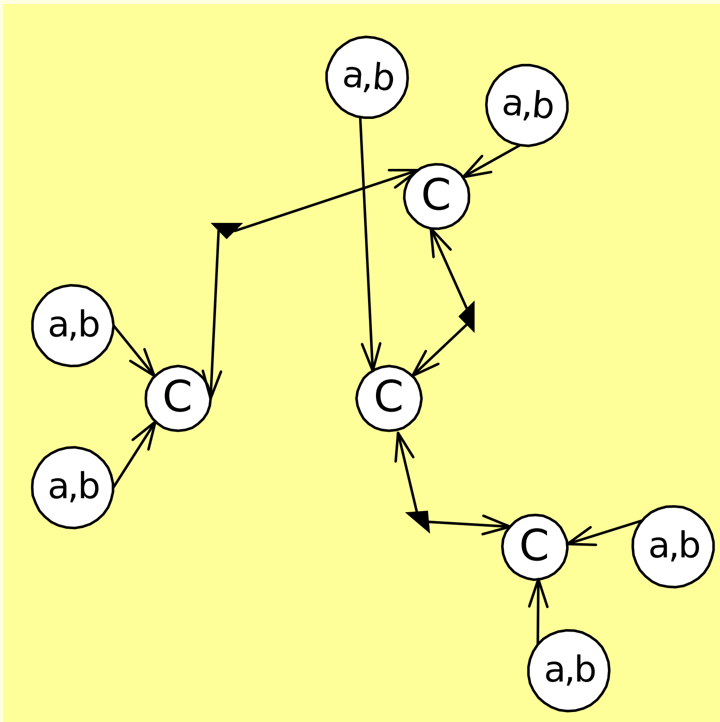
Evaluate D^

$\hat{D} =$ 

$=$ 

Evaluate D^

$\hat{D} =$



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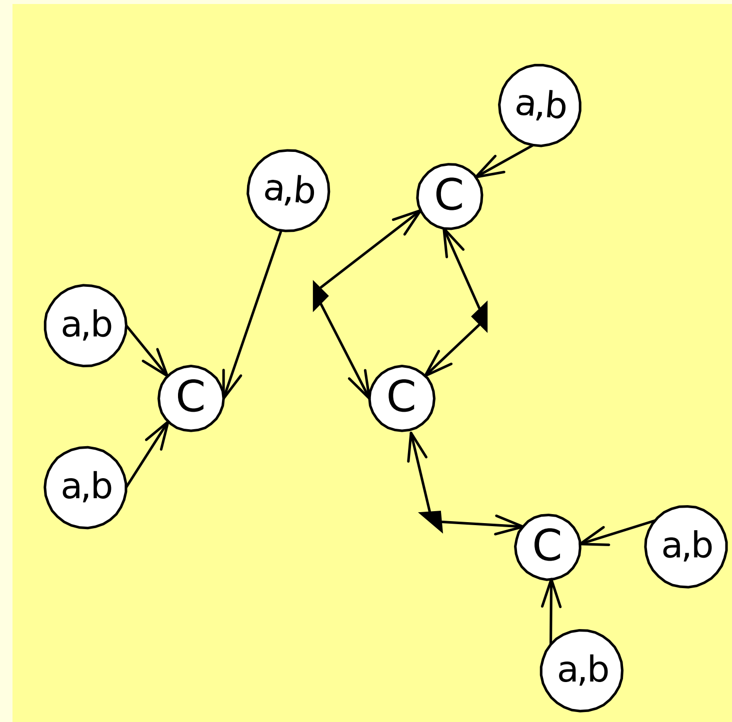
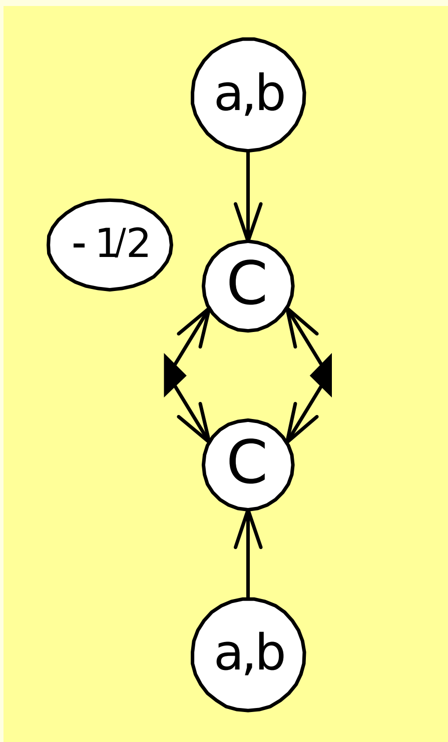


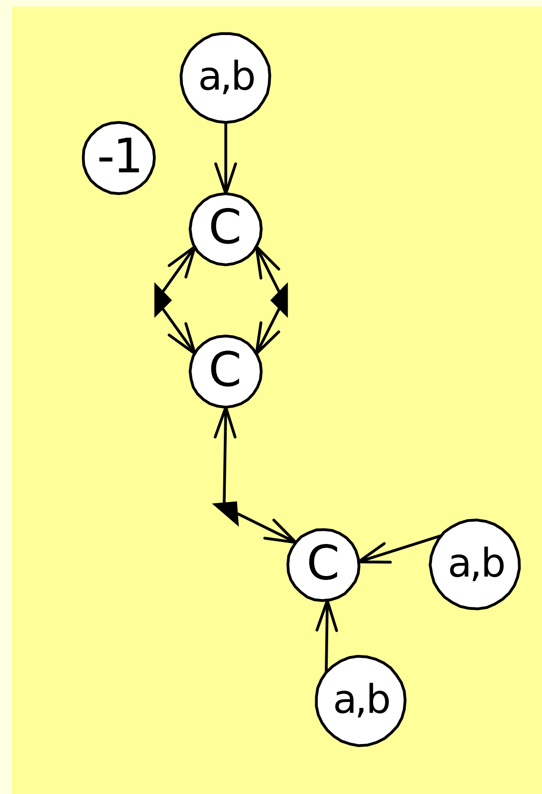
Transformed Cubic

$$\hat{A}\hat{x}^3 + 3\hat{C}\hat{x}\hat{w}^2 + \hat{D}\hat{w}^3 = \hat{A}\left(\hat{x}^3 + 3\tilde{C}\hat{x}\hat{w}^2 + \tilde{D}\hat{w}^3\right)$$

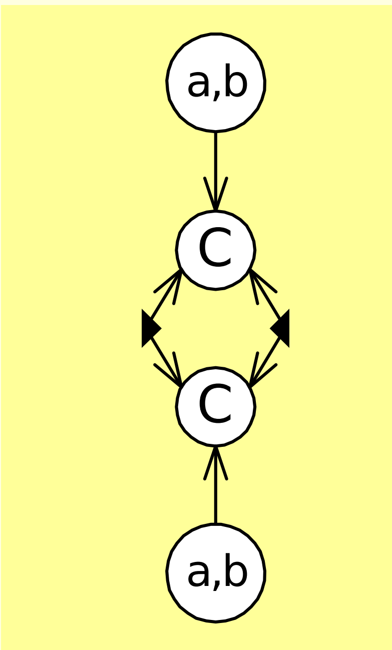
$\tilde{C} =$



$\tilde{D} =$



An Interesting Choice for a,b



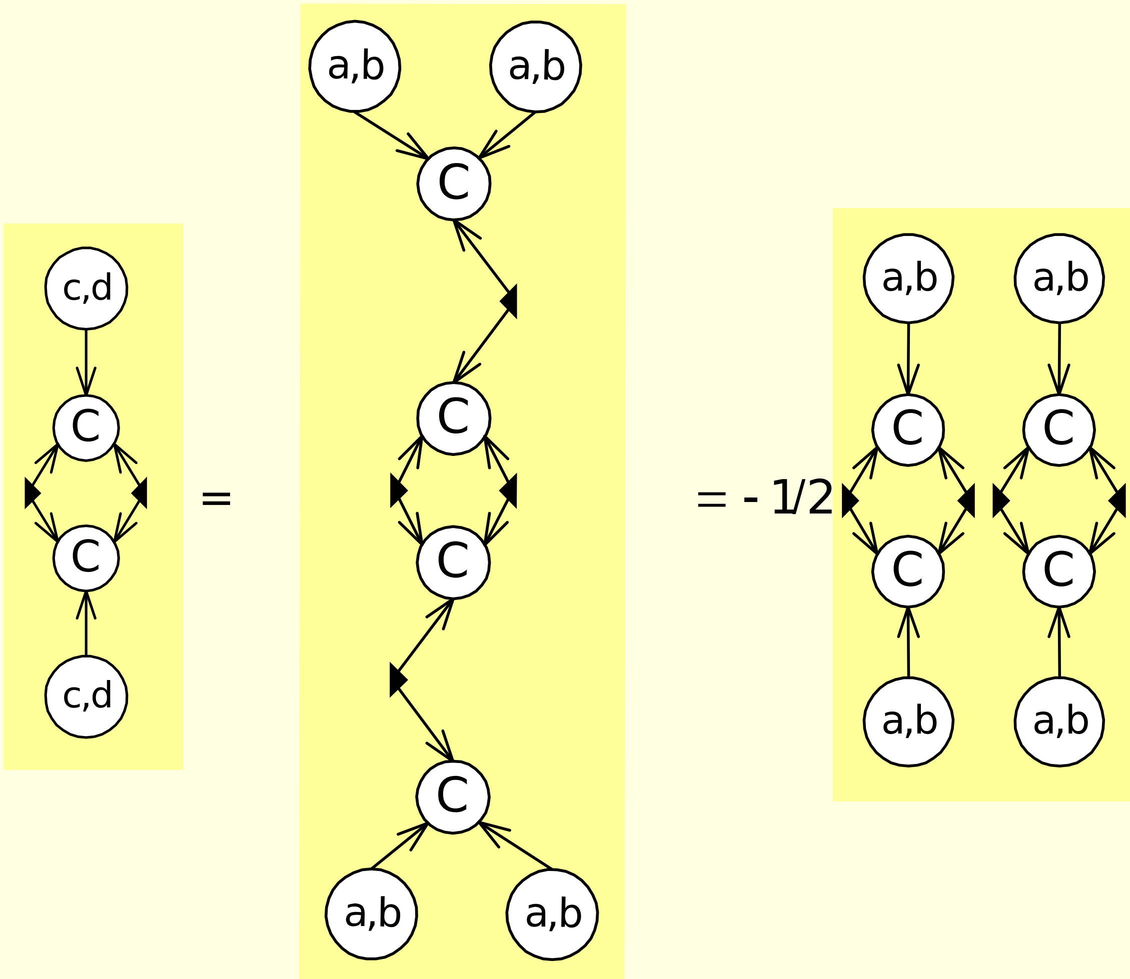
$= 0$

$$\hat{A}\left(\hat{x}^3 + 3\tilde{C}\hat{x}\hat{w}^2 + \tilde{D}\hat{w}^3\right) = 0$$

ß

$$\hat{x}^3 + \tilde{D}\hat{w}^3 = 0$$

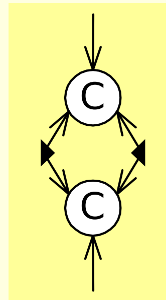
Implications for value of c,d



Solution

1) Find Roots of

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

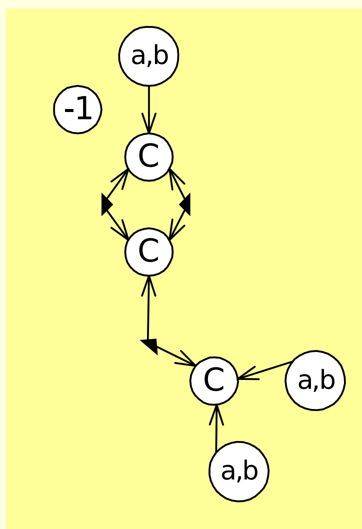


3) Solve for x^

$$\hat{x}^3 + \Delta\hat{w}^3 = 0$$

2) Calculate

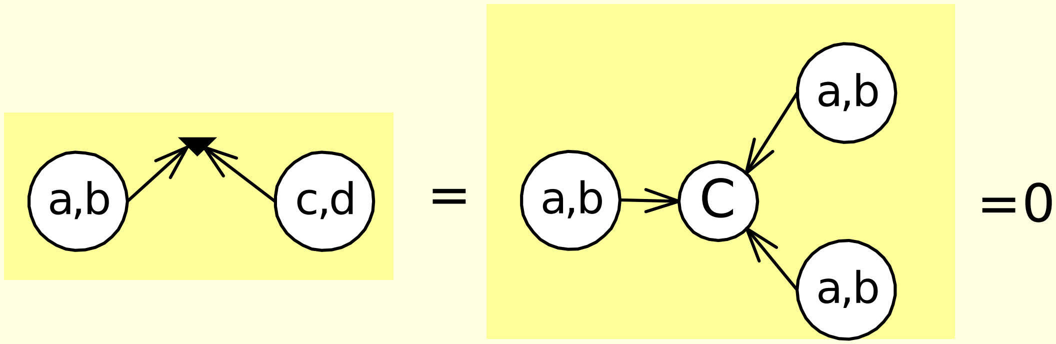
$$\Delta =$$



4) Transform back via

$$[x \quad w] = [\hat{x} \quad \hat{w}] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

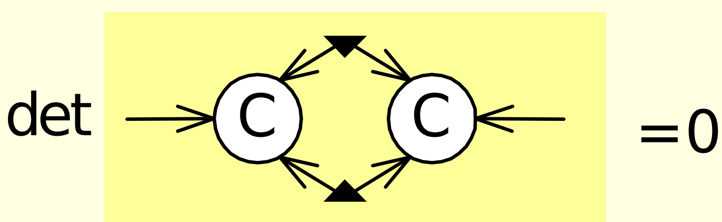
Only Time This Won’t Work

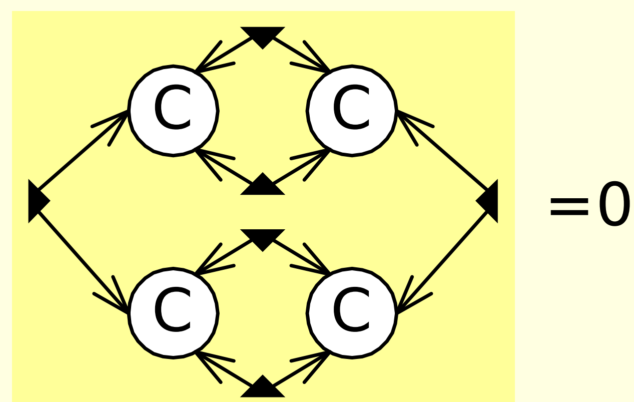


(a,b)=(c,d) is a double root of Quadratic

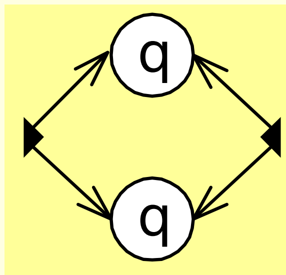
(a,b) is a root of C

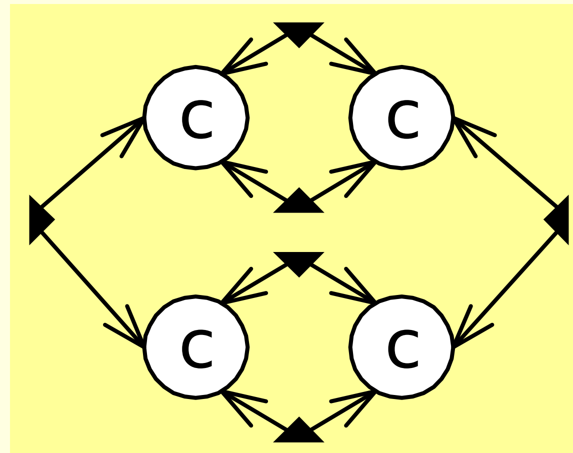
It is a double root of C

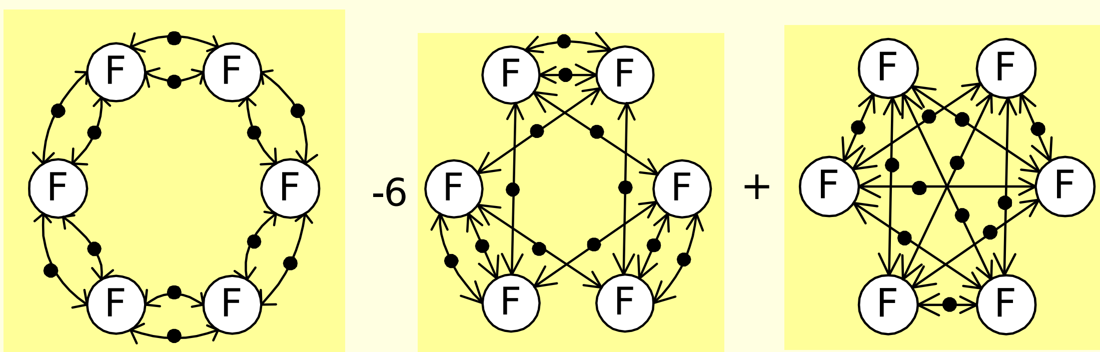




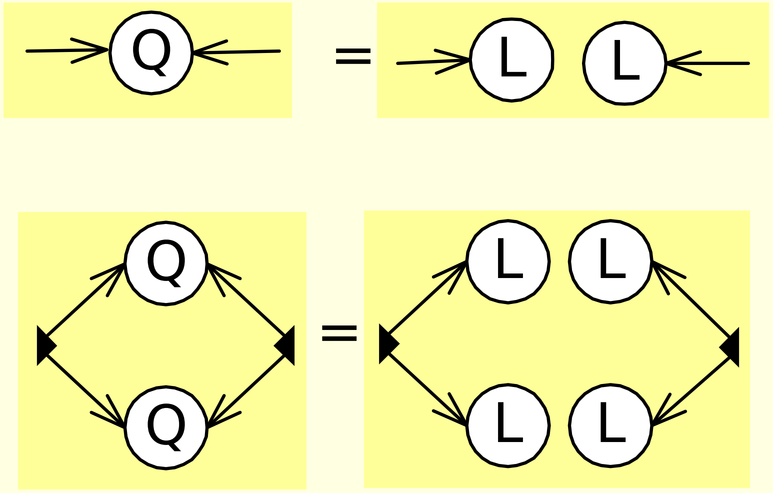
1DH Discriminants



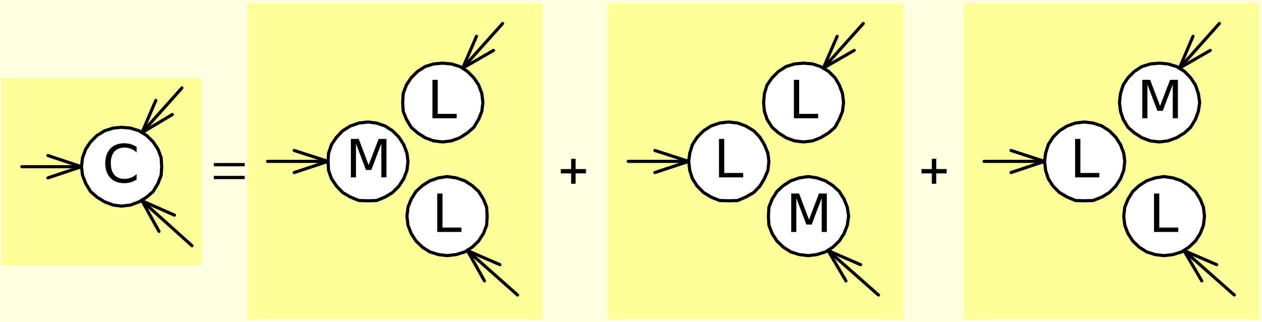


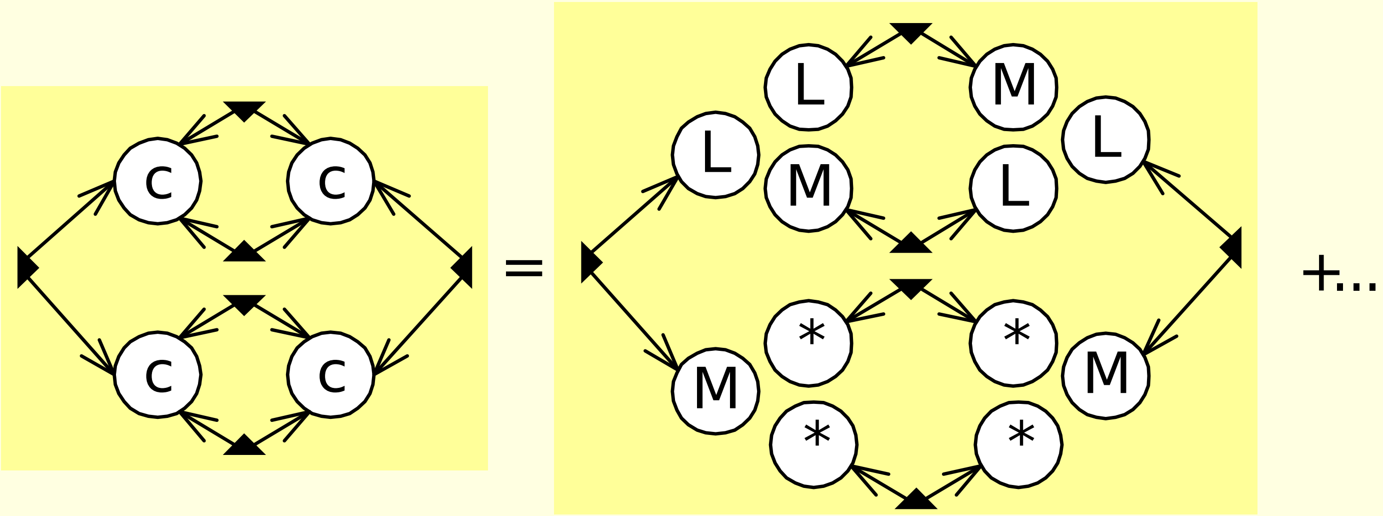


Why Discriminants Work

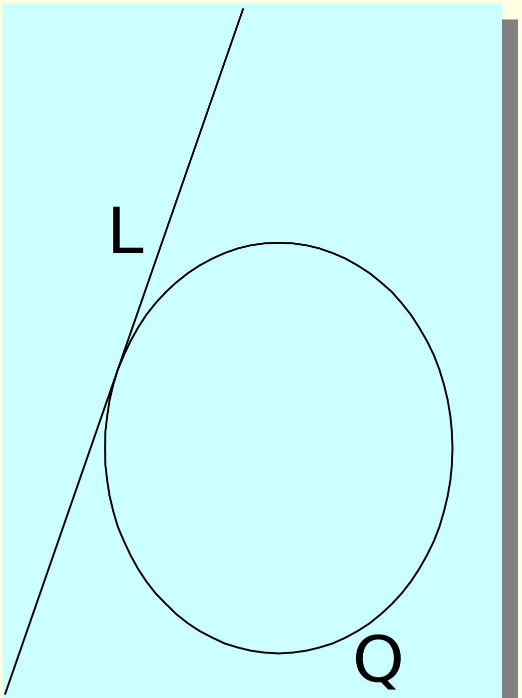


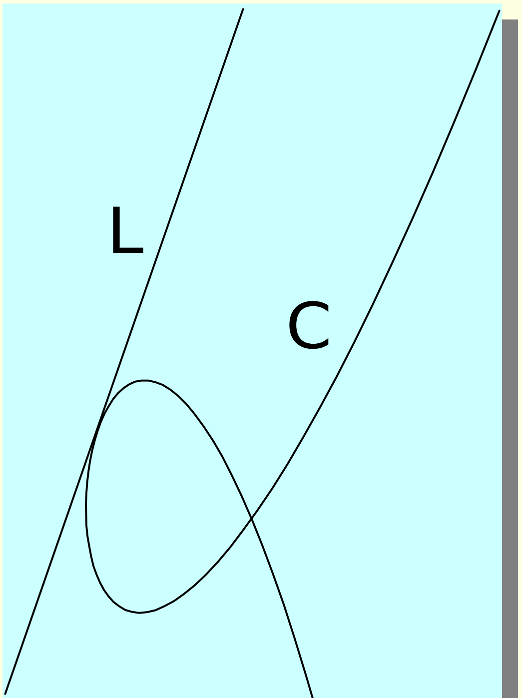
Why Discriminants Work





Tangency



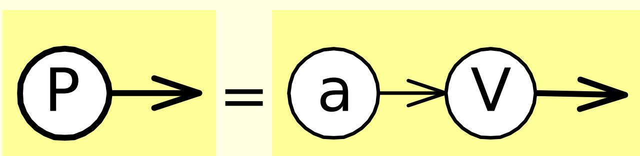


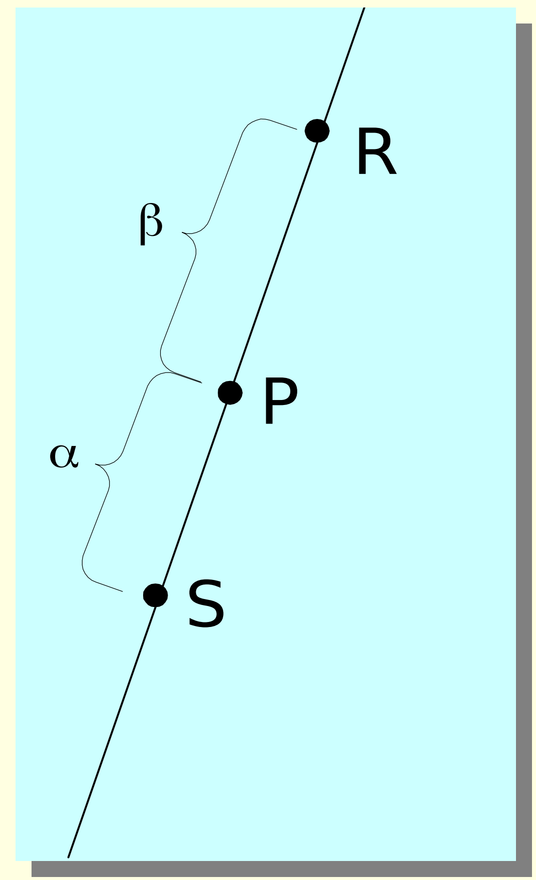
Parametrize Line

$$\mathbf{P}(a,b) = a\mathbf{R} + b\mathbf{S}$$

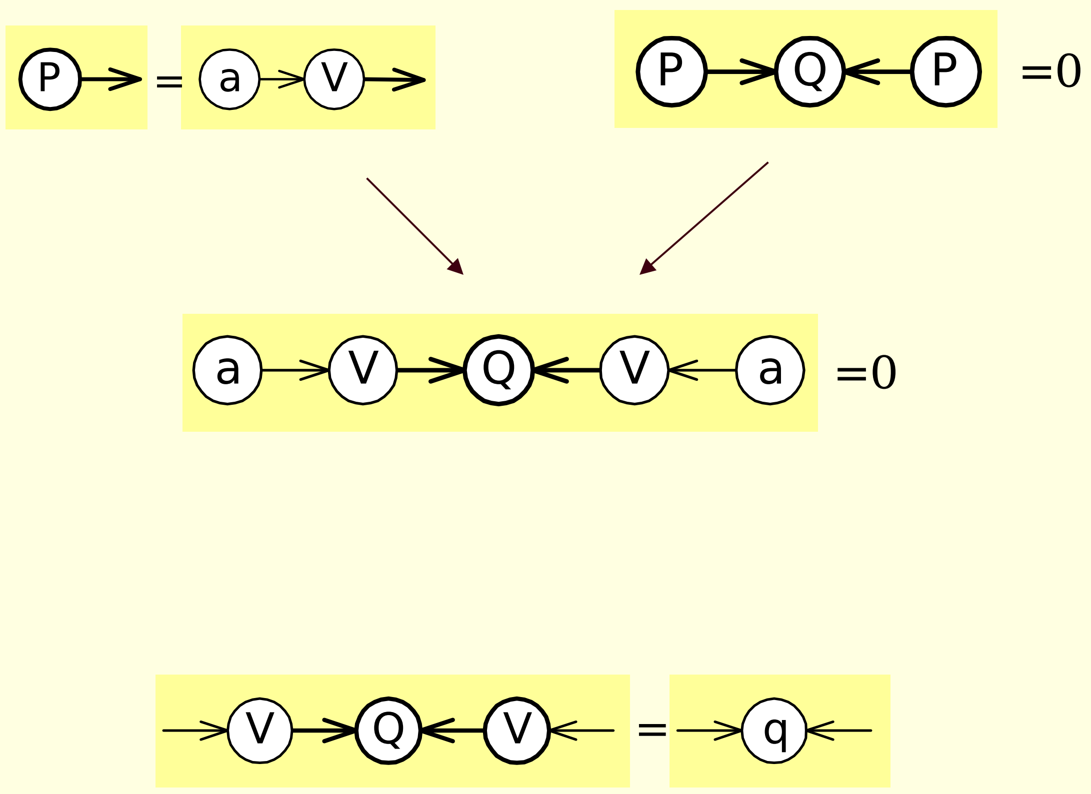
$$\mathbf{P} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix}$$

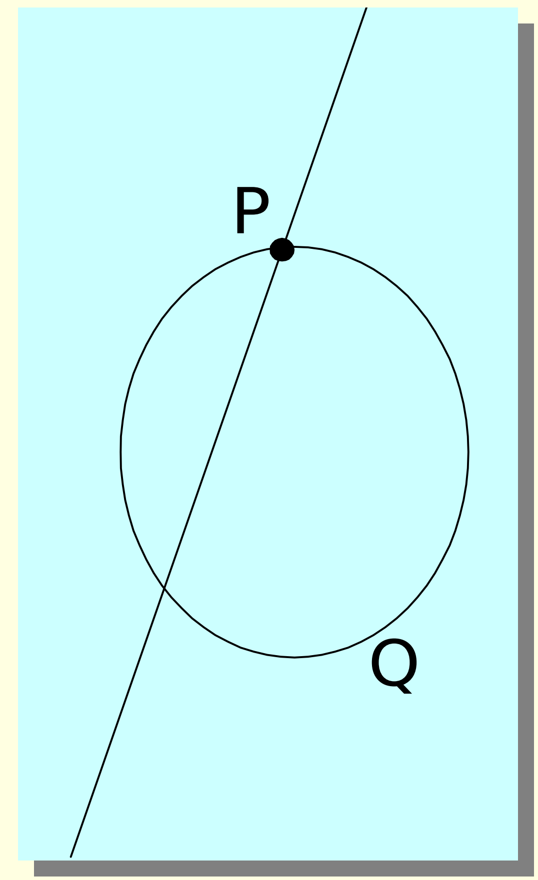
$$\mathbf{P} = \mathbf{aV}$$



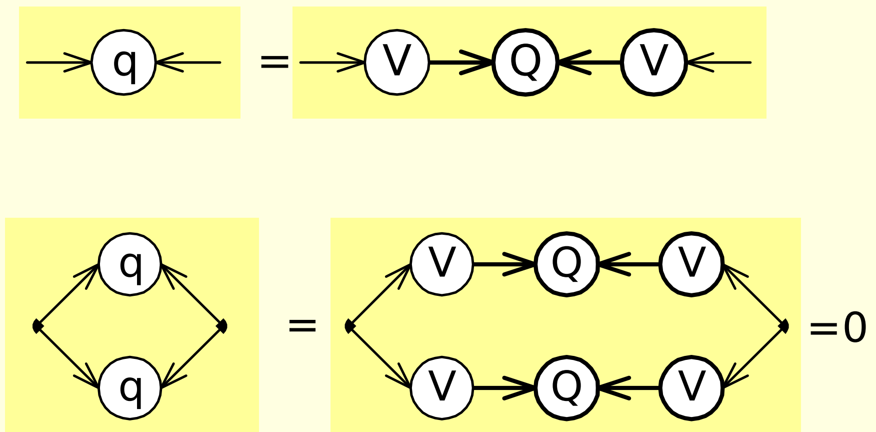


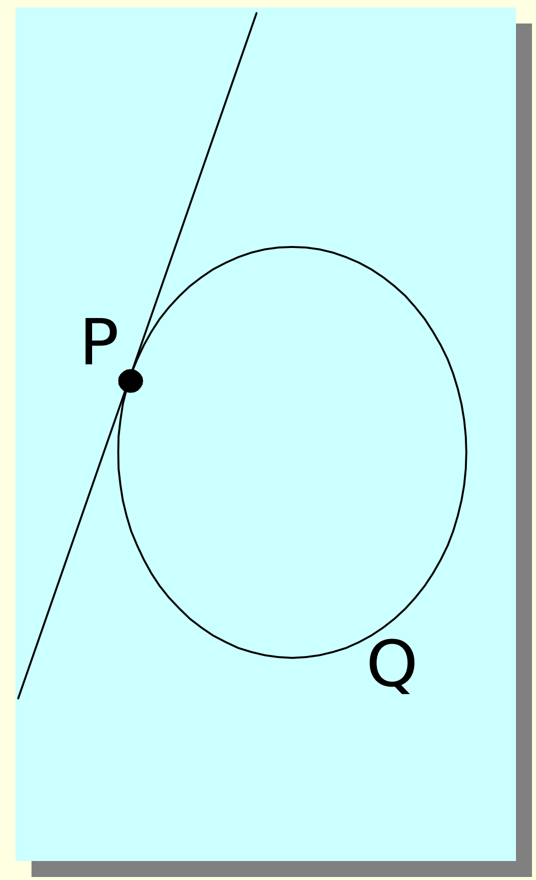
Points on Line And Quadric





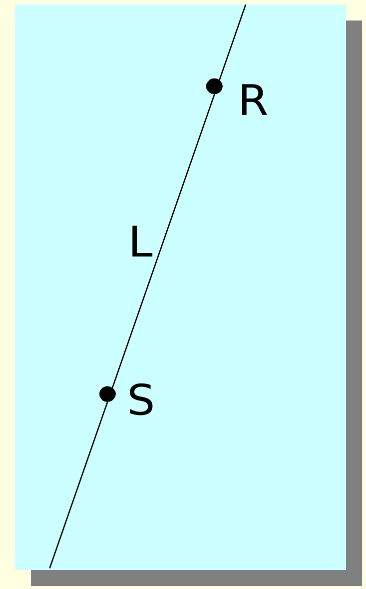
Double Roots Mean Tangent

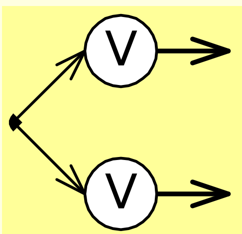




Reinterpret Diagram Fragment

$$\mathbf{V} = \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

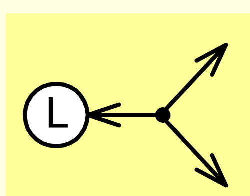




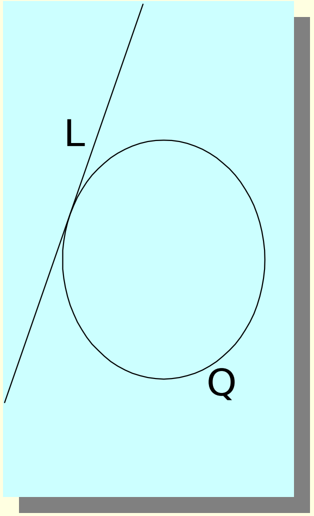
$$= \begin{bmatrix} R^1 & S^1 \\ R^2 & S^2 \\ R^3 & S^3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} R^1 & R^2 & R^3 \\ S^1 & S^2 & S^3 \end{bmatrix}$$

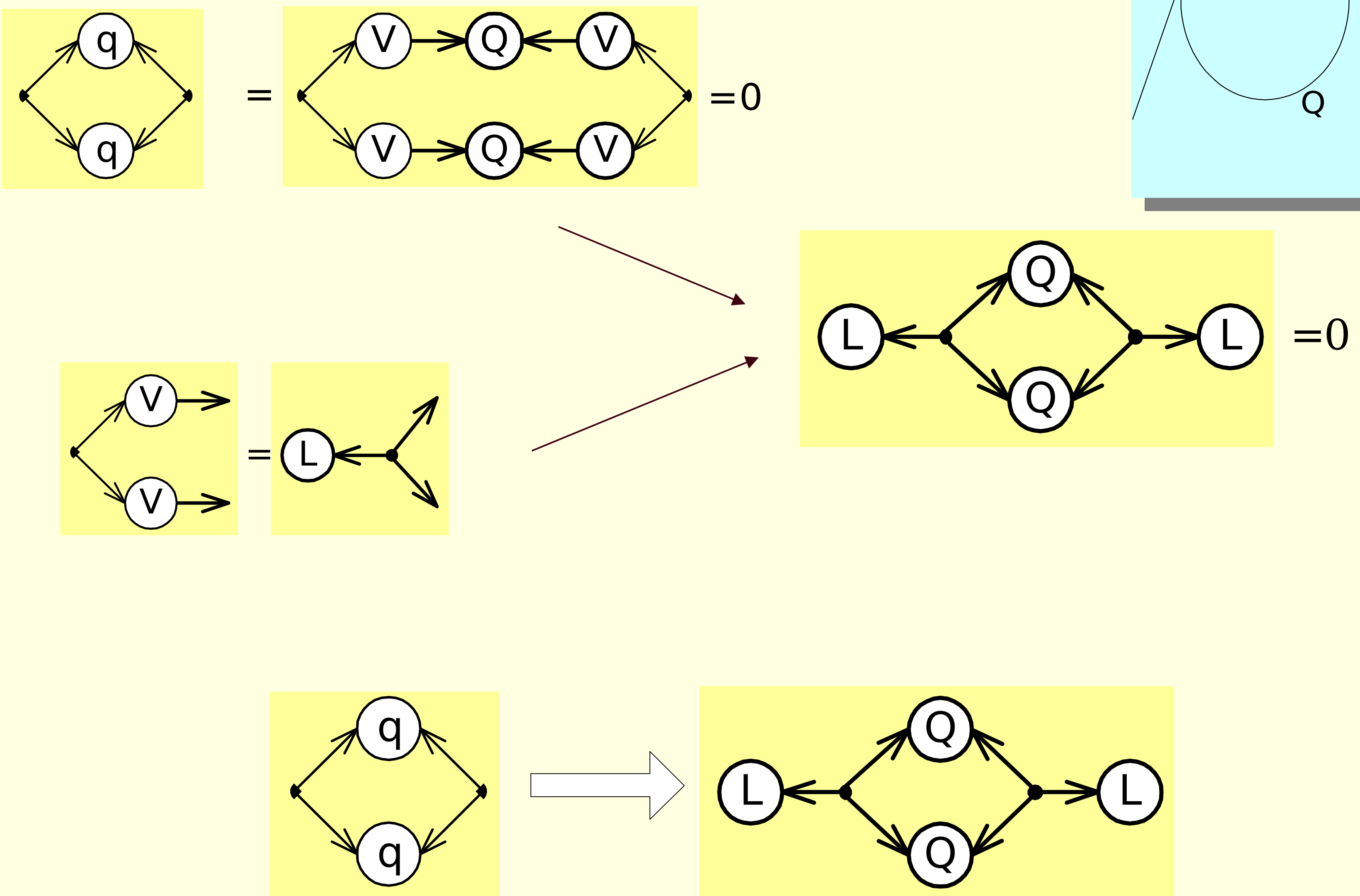
$$= \begin{bmatrix} 0 & R^1S^2 - R^2S^1 & R^1S^3 - R^3S^1 \\ R^2S^1 - R^1S^2 & 0 & R^2S^3 - R^3S^2 \\ R^3S^1 - R^1S^3 & R^3S^2 - R^2S^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{bmatrix}$$

$=$ 

Line Tangent To Quadric





Line Tangent to Cubic